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LETTERS TO THE EDITOR

PERIODIC SOLUTIONS OF THE RELATIVISTIC HARMONIC OSCILLATOR

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Arguments have been given by Greenspan [1] to suggest that the equation of motion for a relativistic harmonic oscillator is

$$\ddot{x} + (1 - \dot{x}^2)^{3/2} x = 0. \tag{1}$$

This normalized, dimensionaless form of the equation is based on taking the rest mass to be unity and the maximum speed of signal transmission (the speed of light) to also be unity. While Greenspan's interest was in constructing a scheme to provide numerical solutions for equation (1), the purposes of this note are to show that all the solutions to the relativistic harmonic oscillator are periodic and determine a method for calculating analytic approximations to its solutions.

Introducing the phase space variables (x, y), equation (1) can be written in the system form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -(1 - y^2)^{3/2} x. \tag{2a, b}$$

Consequently, the trajectories in phase space are given by solutions to the first order, ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{(1 - y^2)^{3/2} x}{y}. \tag{3}$$

Observe that since the physical solution of either equation (1) or equation (3) are real, the phase space has a "strip" structure, i.e.,

$$-\infty < x < +\infty, \qquad -1 < y < +1. \tag{4}$$

In other words, unlike the usual non-relativistic harmonic oscillator, the relativistic oscillator is bounded in the y variable. A non-linear oscillator equation of motion, also having a "strip" structure, has been investigated by Mickens [2]. However, for that case, the "strip" is such that the y variable can be unbounded, while the x variable is bounded.

The equation (3) is invariant under the following three co-ordinate transformations:

$$S_1\colon x \to -x, \quad y \to y \qquad \text{(reflection in the } y\text{-axis)},$$

$$S_2\colon x \to x, \quad y \to -y \qquad \text{(reflection in the } x\text{-axis)},$$

$$S_3\colon x \to -x, \quad y \to -y \qquad \text{(inversion through the origin)}.$$

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Likewise, the null-clines [3], curves along which the slope dy/dx is either zero or unbounded, are given by

$$dy/dx = 0, \qquad x = 0 \text{ or the } y\text{-axis} \tag{5a}$$

$$dy/dx = \infty, \qquad y = 0 \text{ or the } x\text{-axis} \tag{5b}$$

Using exactly the same arguments as Mickens and Semwogerere [4], it follows that all the trajectories to equation (3) are closed in the open region of phase space given by equation (4). This implies that all the physical solutions to equation (1) are periodic.

The method of harmonic balance [5] can now be applied to obtain analytic approximations to the periodic solutions of equation (1). To proceed, we make a change of variable, $y \to w$, such that

$$-\infty < w < \infty. \tag{6}$$

The required transformation is [2]

$$y = \frac{w}{\sqrt{1 + w^2}}. \tag{7}$$

It follows that

$$\frac{dx}{dt} = y = \frac{w}{\sqrt{1 + w^2}}, \qquad \frac{dy}{dt} = \frac{1}{(1 + w^2)^{3/2}} \frac{dw}{dt}. \tag{8, 9}$$

Substituting, on the right side of equation (2b), the expression of equation (7) for y, equating this result to the right side of equation (9), and solving for dw/dt gives

$$dw/dt = -x. \tag{10}$$

The corresponding second order differential equation for w is

$$\frac{d^2 w}{dt^2} = -\frac{dx}{dt}, \tag{11}$$

or using equation (8),

$$\frac{d^2 w}{dt^2} + \frac{w}{\sqrt{1 + w^2}} = 0. \tag{12}$$

Inspection of this equation shows that in the variable w, equation (1) is a conservative oscillator.

The method of first harmonic balance [5] can now be applied to equation (12). Let $w_1(t)$,

$$w_1 = A \cos \omega t, \tag{13}$$

where ω is to be determined as a function of A, and the initial conditions to equation (12) are taken as

$$w(0) = A, \qquad dw(0)/dt = 0. \tag{14}$$

Substituting w_1 into equation (12) and applying harmonic balancing gives

$$\omega(A) = \frac{1}{[1 + A^2/2]^{1/4}}, \tag{15}$$

and

$$w_1(t) = A \cos \left\{ \frac{t}{[1 + A^2/2]^{1/4}} \right\}. \tag{16}$$

The corresponding approximation to y is gotten from equation (7):

$$y_1(t) = \frac{A \cos \omega t}{\sqrt{(1 + A^2/2) + (A^2/2) \cos 2\omega t}}. \tag{17}$$

Likewise, $x_1(t)$ can be calculated by integrating equation (2a) subject to the restrictions

$$x_1(0) = 0, \qquad \frac{dx_1(0)}{dt} = \frac{A}{\sqrt{1 + A^2}}. \tag{18}$$

This integration can be easily done to give [6]

$$\omega x_1(t) = \sin^{-1}\left[\sqrt{\frac{A^2}{1 + A^2}} \sin \omega(t)\right]. \tag{19}$$

Using expansion [6], $z^2 < 1$,

$$\sin^{-1}(z) = z + \frac{1}{2 \cdot 3} z^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} z^5 + \ldots, \tag{20}$$

it is evident that the first approximation to x, namely $x_1(t)$, contains contributions from all the odd multiples of ω. Define $\alpha(A)$ to be

$$\alpha(A) = \frac{A}{\sqrt{1 + A^2}}, \tag{21a}$$

where

$$0 < \alpha(A) < 1, \qquad A > 0. \tag{21b}$$

It can be shown, using equations (19) and (20) that $x_1(t)$ has the representation

$$\omega(A)x_1(t) = \alpha\left[1 + \left(\frac{1}{8}\right)\alpha^2 + \left(\frac{3}{64}\right)\alpha^4\right] \sin(\omega t)$$

$$-\left(\frac{\alpha^3}{24}\right)\left[1 + \left(\frac{3}{128}\right)\alpha^2\right] \sin(3\omega t) + \left(\frac{3\alpha^5}{640}\right) \sin(5\omega t) + O(\alpha^7). \tag{22}$$

In summary, all the solutions of the relativistic harmonic oscillator are periodic. This was demonstrated using the geometrical properties of the corresponding phase space variables. However, unlike the usual (non-relativistic) harmonic oscillator, the relativisitic oscillator contains higher-order multiples of the fundamental ω. A first approximation to the periodic solutions was calculated by transforming to a new set of variables, $(x, y) \rightarrow (x, w)$, and then applying the method of harmonic balance to the resulting differential equations. This note and a previous one [2] allow the following conclusion to be reached: oscillatory systems having "strip" type phase space diagrams can be analyzed in exactly the same manner as systems requiring the complete phase space provided that the proper transformation to new variables is made.

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REFERENCES

1. D. GREENSPAN 1997 *Particle Modeling*. Boston, Massachusetts: Birkhäuser. See section 17.9.
2. R. E. MICKENS 1997 *Journal of Sound and Vibration* (accepted for publication). Regulation of singular ordinary differential equations modelling oscillating systems.
3. L. EDELSTEIN-KESHET 1988 *Mathematical Models in Biology*. New York: McGraw-Hill. See Chapter 5.
4. R. E. MICKENS and D. SEMWOGERERE 1997 *Journal of Sound and Vibration* **204**, 556–559. A phase space analysis of a non-linear oscillator equation.
5. R. E. MICKENS 1996 *Oscillations in Planar Dynamic Systems*. Singapore: World Scientific. See sections 4.1 and 4.2.
6. I. S. GRADSHTEYN and I. M. RYZHIK 1980 *Table of Integrals, Series, and Products*. New York: Academic Press.